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Exact solutions for two nonlinear equations: I

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Abstract. In this paper we shall present a certain type of exact solutions for the two nonlinear equations by making an ansatz for the solution.

As we all know, nonlinear phenomena are very important in various fields of science, especially in physical sciences. One must solve nonlinear equations to get a knowledge of the system [1, 2], but the methods are very few and have some constraints. The inverse scattering method [3] and the Hirota method [1] are two successful methods, but in some cases [1, 2], they cannot do very well. Here we shall make use of the property of hyperbolic functions to obtain exact solutions directly.

The first equation we deal with is as follows [1]:

$$U_t + \alpha U U_x + \beta U_{xx} + \delta U_{xxx} + \gamma U_{xxxx} = 0. \tag{1}$$

Equation (1) can be seen as the extended Burgers equation, where α, β, δ and γ are parameters. In [1], Kazuhiro has written it in Hirota's trilinear form, but he could not give the explicit solution of equation (1).

Here we look for the travelling solutions of equation (1), that is we assume that

$$U(x, t) = U(x - \lambda t) \equiv U(\xi) \tag{2}$$

where λ is the velocity to be determined.

Integrating equation (1) and taking equation (2) into consideration, we get

$$-\lambda U + \tfrac{1}{2}\alpha U^2 + \beta U_\xi + \delta U_{\xi\xi} + \gamma U_{\xi\xi\xi} + C_0 = 0 \tag{3}$$

where C_0 is the integration constant, which we take to be zero.

According to the relation of the derivative of the hyperbolic function and the function itself, we know that the nonlinear term and the derivative term can be balanced for some appropriate algebraic combination of the hyperbolic function. To the equation (1), we make the ansatz

$$U = \sum_{i=0}^{m} a_i (\tanh \mu\xi)^i \tag{4}$$

where the integer m, $a_i (i = 1, \ldots, m)$ and μ are parameters to be determined.

The requirement that the highest power of the function $\tanh(\mu\xi)$ for the nonlinear term $\frac{1}{2}\alpha u^2$ and that for the derivative term $\gamma u_{\xi\xi\xi}$ must be equal gives the following relation:

$$2m = m + 3. \tag{5}$$

Thus we know that $m = 3$ and equation (4) can be written as follows:

$$U = a_0 + a_1 \tanh \mu\xi + a_2 \tanh^2 \mu\xi + a_3 \tanh^3 \mu\xi. \tag{6}$$

Inserting equation (6) into equation (3), we get the following parametric equations by equating the same power of $\tanh \mu\xi$:

$$60a_3\gamma\mu^2 - \tfrac{1}{2}\alpha a_3^2 = 0 \tag{7}$$

$$24a_2\gamma\mu^3 - 12\delta\mu^2 a^3 - \alpha a_3 a_2 = 0 \tag{8}$$

$$-a_3(114\gamma\mu^3 - 3\beta\mu) + 6\gamma\mu^3 a_1 = 6\delta\mu^2 a_2 + \tfrac{1}{2}\alpha(2a_1a_3 + a_2^2) \tag{9}$$

$$18\delta\mu^2 a_3 - 2\delta\mu^2 a_1 - a_2(40\gamma\mu^3 - 2\beta\mu) = \tfrac{1}{2}\alpha(2a_1a_2 + 2a_0a_3) - \lambda a_3 \tag{10}$$

$$a_3(60\gamma\mu^3 - 3\beta\mu) + 8\delta\mu^2 a_2 - 8\gamma\mu^3 a_1 = \tfrac{1}{2}\alpha(a_1^2 + 2a_0a_2) - \lambda a_2 - \beta\mu a_1 \tag{11}$$

$$-6\delta\mu^2 a_3 + (16\gamma\mu^3 - 2\beta\mu)a_2 + 2\delta\mu^2 a_1 - \alpha a_0 a_1 + \lambda a_1 = 0 \tag{12a}$$

$$6\gamma\mu^3 a_3 + 2\delta\mu^2 a_2 - (2\gamma\mu^3 - \beta\mu)a_1 + \tfrac{1}{2}\alpha a_0^2 - \lambda a_0 = 0. \tag{12b}$$

From equations (7)-(12), we get

$$a_3 = 120\gamma\mu^3/\alpha \tag{13}$$

$$a_2 = -15\delta\mu^2/\alpha \tag{14}$$

$$a_1 = (120\times 114\gamma\mu^3 - 360\beta\mu + 22.5\delta^2\mu/\gamma)/(-114\alpha) \tag{15}$$

$$\begin{aligned} a_0 = \frac{2}{\alpha}\left(2\delta\mu^2 - \frac{30\times 114\mu^2\beta\gamma - 960\times 114\delta\gamma^2\mu^4}{120\times 114\gamma^2\mu^2 - 360\beta\gamma + 22.5\delta^2}\right) \\ \pm\left\{\left(2\delta\mu^2 - \frac{30\times 114\gamma\beta\mu^2 - 960\times 114\delta\gamma^2\mu^4}{120\times 114\gamma^2\mu^2 - 360\beta\gamma + 22.5\delta^2}\right)^2\right. \\ -2\left[720\gamma^2\mu^6 - 30\delta^2\mu^4 + \frac{1}{114}(2\gamma\mu^3 - \beta\mu)\right. \\ \left.\left.\times\left(120\times 114\gamma\mu^3 - 360\beta\mu + \frac{22.5\delta^2\mu}{\gamma}\right)\right]\right\}^{1/2}\frac{2}{\alpha} \end{aligned} \tag{16}$$

$$\mu^2 = \left(\frac{3780}{114}\beta^2\gamma^2 - \frac{1113.75}{114}\delta^2\beta\gamma + \frac{54.84375\delta^4}{114}\right)(1800\beta\gamma^3 - 112.5\gamma^2\delta^2)^{-1} \tag{17}$$

$$\lambda = \alpha a_0 - 2\delta\mu^2 + \frac{30\times 114\delta\mu^2\beta\gamma - 960\times 114\delta\gamma^2\mu^4}{120\times 114\gamma^2\mu^2 - 360\beta\gamma + 22.5\delta^2} \tag{18}$$

and one constraint equation for the parameters β, γ and δ:

$$\begin{aligned} A_1\delta^{12} + B_1\beta\gamma\delta^{10} + C_1\gamma^2\delta^{10} + D_1\gamma^4\delta^8 + E_1\gamma^2\beta^2\delta^8 + F_1\beta\gamma^3\delta^8 \\ + G_1\gamma^3\beta^3\delta^6 + H_1\beta\gamma^3\delta^6 + I_1\beta^2\gamma^4\delta^6 + J_1\beta^4\gamma^4\delta^4 + K_1\beta^2\gamma^6\delta^4 \\ + L_1\beta^3\gamma^5\delta^4 + M_1\beta^4\gamma^6\delta^2 + N_1\beta^3\gamma^7\delta^2 + P_1\beta^5\gamma^7 + Q_1\beta^4\gamma^8 = 0. \end{aligned} \tag{19}$$

Here $A_1, B_1, \ldots, Q_1$ are given as follows:

$$A_1 = -IMB + QM^2 \tag{20}$$

$$B_1 = 2QLM + I(MA - LB) \tag{21}$$

$$C_1 = SM^2 - J \tag{22}$$

$$D_1 = -PM^2 - HMB \tag{23}$$

$$E_1 = Q(L^2 + 2KM) + I(LA - KB) \tag{24}$$

$$F_1 = -B^2E + J(MA - LB) - GMB + NM^2 + 2SLM \tag{25}$$

$$G_1 = IKA + 2QKL \tag{26}$$

$$H_1 = H(MA - LB) - 2PLM \tag{27}$$

$$J_1 = B^2D + 2ABE + G(MA - LB) + J(LA - KB) - FMB + 2NLM + S(L^2 + 2KM) \tag{28}$$

$$J_1 = QK^2 \tag{29}$$

$$K_1 = H(LA - KB) - P(L^2 + 2KM) \tag{30}$$

$$L_1 = 2SKL + N(L^2 + 2KM) + JKA + G(LA - KB) + F(MA - LB) - A^2E + B^2C - 2ABD \tag{31}$$

$$M_1 = A^2D - 2ABC + F(LA - KB) + GKA + 2KLN + SK^2 \tag{32}$$

$$N_1 = HKA - 2PKL \tag{33}$$

$$P_1 = A^2C + FAK + NK^2 \tag{34}$$

$$Q_1 = -PK^2. \tag{35}$$

The quantities $A, B, \ldots, S$ are numerals, which are given as follows:

$$A = 1800 \qquad B = 112.5 \tag{36}$$

$$C = \frac{-360 \times 23\,760}{114^2} \qquad D = \frac{25\,272\,000}{114^2} \tag{37}$$

$$E = \frac{215\,662.5}{114^2} \qquad F = \frac{-2\,332\,800}{114} \tag{38}$$

$$G = 907\,200 \qquad H = -21\,600 \tag{39}$$

$$I = \frac{22.5^3}{2 \times 114^2} \qquad J = \frac{22.5 \times 12\,780}{114} \tag{40}$$

$$K = \frac{3780}{114} \qquad L = \frac{-1113.75}{114} \tag{41}$$

$$M = \frac{54.843\,75}{114} \qquad N = 2\,073\,600 \tag{42}$$

$$P = 13\,132\,300 \qquad Q = \frac{60 \times 22.5^2}{114} \tag{43}$$

$$S = 1\,379\,700. \tag{44}$$

Thus we obtain one exact solution of equation (1) that cannot be obtained by Hirota's method in [1].

The second equation we deal with is from [2], and it can be written as follows:

$$A_\xi^2[2H\tau A^2(1-\tfrac{1}{2}A^2)] = \gamma_0 A^2[1+2\tau/\alpha^2\gamma_0 - V^2/4\gamma_0 - (A^2/2)(1+4\tau/\gamma_0\alpha^2)+(\tau/2\gamma_0\alpha^2)A^4]. \tag{45}$$

In [2], one exact solution for the case $\tau=0$, when $\tau\neq 0$, was found and no exact solution of equation (45). Making use of the property of the hyperbolic function, we obtain one polariton solution in an analytic form for the case $\tau\neq 0$.

Let

$$A^2 = B. \tag{46}$$

Then equation (45) will become

$$B_\xi^2(1+2\tau B-\tau B^2) = 4\gamma_0 B^2\left(1+\frac{2\tau}{\alpha^2\gamma_0}-\frac{V^2}{4\gamma_0}\right) - 2\gamma_0 B^3\left(1+\frac{4\tau}{\gamma_0\alpha^2}\right)+\frac{2\tau}{\alpha^2}B^4 \tag{47}$$

then we assume that

$$B = \frac{1}{a+b\cosh\mu\xi} \tag{48}$$

where a, b and μ are the parameters to be determined. Inserting equation (48) into equation (47), equating the same power of $\cosh\mu\xi$, we get the following parametric equations:

$$b^4(4\gamma_0+8\tau/\alpha^2-V^2) = b^2\mu^2 \tag{49}$$

$$b^2\mu^2(2ab+2\tau b) = \left(4\gamma_0+\frac{8\tau}{\alpha^2}-V^2\right)4ab^3 - \left(2\gamma_0+\frac{8\tau}{\alpha^2}\right)b^3 \tag{50}$$

$$b^2\mu^2(a^2-\tau+2a\tau-b^2) = \left(4\gamma_0+\frac{8\tau}{\alpha^2}-V^2\right)6a^2b^2+\frac{2\tau}{\alpha^2}b^2-\left(2\gamma_0+\frac{8\tau}{\alpha^2}\right)3ab^2 \tag{51}$$

$$-b^2\mu^2(2\tau b+2ab) = \left(4\gamma_0+\frac{8\tau}{\alpha^2}-V^2\right)4a^3b+\frac{4\tau ab}{\alpha^2}-\left(2\gamma_0+\frac{8\tau}{\alpha^2}\right)3a^2b \tag{52}$$

$$-b^2\mu^2(a^2-\tau+2a\tau) = \left(4\gamma_0+\frac{8\tau}{\alpha^2}-V^2\right)a^4+\frac{2\tau a^2}{\alpha^2}-\left(2\gamma_0+\frac{8\tau}{\alpha^2}\right)a^3. \tag{53}$$

From equation (49) to equation (53), we have

$$\mu^2 = 4\gamma_0+8\tau/\alpha^2-V^2 \tag{54}$$

$$a = \tau+\frac{1}{\mu^2}\left(\gamma_0+\frac{4\tau}{\alpha^2}\right) \tag{55}$$

$$b^2 = -5a^2+2\tau a+\left(\gamma_0+\frac{4\tau}{\alpha^2}\right)\frac{6a}{\mu^2}-\frac{2\tau}{\alpha^2\mu^2}-\tau \tag{56}$$

and the two constraint equations for the parameters τ, γ_0, V and α

$$\mu^2(2\tau a-5a^2-\tau)-\frac{2\tau}{\alpha^2}+3\left(2\gamma_0+\frac{8\tau}{\alpha^2}\right)a = \left[3\left(2\gamma_0+\frac{8\tau}{\alpha^2}\right)a^2-\frac{4\tau a}{\alpha^2}-4a^3\mu^2\right][2(a+\tau)]^{-1} \tag{57}$$

$$\mu^2(2\tau a-5a^2-\tau)-\frac{2\tau}{\alpha^2}+3\left(2\gamma_0+\frac{8\tau}{\alpha^2}\right)a$$
$$=\left[\mu^2a^4+2\tau-\left(2\gamma_0+\frac{8\tau}{\alpha^2}\right)a^3\right](\tau-a^2-2\tau a)^{-1}. \tag{58}$$

Therefore, we obtain an exact solution of equation (47). In fact the 'bell' form solution of (48) is just a polariton. We show this as follows:

$$a=(1+\tanh^2\tfrac{1}{2}\mu\xi)/4\beta\tanh\tfrac{1}{2}\mu\xi_0 \tag{59}$$

$$b=1/2\beta\sinh\mu\xi. \tag{60}$$

The solution of equation (48) can then be written as follows:

$$B=\beta[\tanh\tfrac{1}{2}(\xi+\xi_0)-\tanh\tfrac{1}{2}\mu(\xi-\xi_0)] \tag{61}$$

where β and ξ_0 are given as follows:

$$\beta=\pm 1/(4b^2-a^2)^{1/2} \tag{62}$$

$$\xi_0=\mu^{-1}\sinh^{-1}b/2\beta \tag{63}$$

Therefore, we finally obtain

$$A=\pm\sqrt{B}=\pm\left(\frac{1}{a+b\cosh\mu\xi}\right)^{1/2}=\{\beta[\tanh\tfrac{1}{2}\mu(\xi+\xi_0)-\tanh\tfrac{1}{2}\mu(\xi-\xi_0)]\}^{1/2}. \tag{64}$$

When τ tends to zero, one can easily show that the solution of equation (64) comes back to the solution obtained in [2]. Thus we obtain one exact polariton solution for some non-zero τ that is constrained by equations (57) and (58).

Following the same definition as that in [2], we may get the magnetization M and magnetic moment P of the polariton solution of equation (64):

$$M=\frac{S_c}{\alpha}\int A^2\,\mathrm{d}x=\frac{S_c}{\alpha}\int\frac{\mathrm{d}\xi}{a+b\cosh\mu\xi}$$
$$=\begin{cases}\dfrac{S_c}{\alpha}\dfrac{2}{(a^2-b^2)^{1/2}}\ln\dfrac{a+b+(a^2-b^2)^{1/2}}{a+b-(a^2-b^2)^{1/2}} & a^2>b^2\\[2ex] \left(\dfrac{\pi}{\mu(b^2-a^2)^{1/2}}-\dfrac{2}{\mu(b^2-a^2)^{1/2}}\sin^{-1}\dfrac{a}{b}\right)\dfrac{S_c}{\alpha} & b^2>a^2\end{cases} \tag{65}$$

$$P=\frac{S_c}{\alpha}\int A^2\varphi_\xi\,\mathrm{d}\xi=\frac{S_c}{\alpha}\int A^2\frac{V}{2}\frac{1}{1-A^2/2}\,\mathrm{d}\xi$$
$$=\frac{S_c}{\alpha}V\int\frac{\mathrm{d}\xi}{2a-1+2b\cosh\mu\xi}$$
$$=\begin{cases}\dfrac{S_c}{\alpha}V\dfrac{2}{(a_1^2-b_1^2)^{1/2}}\ln\dfrac{a_1+b_1+(a_1^2-b_1^2)^{1/2}}{a_1+b_1-(a_1^2-b_1^2)^{1/2}} & a_1^2>b_1^2\\[2ex] \dfrac{S_c}{\alpha}V\left(\dfrac{\pi}{\mu(b_1^2-a_1^2)^{1/2}}-\dfrac{2}{\mu(b_1^2-a_1^2)^{1/2}}\sin^{-1}\dfrac{a_1}{b_1}\right) & a_1^2<b_1^2\end{cases} \tag{66}$$

where

$$a_1=2a-1 \qquad b_1=2b. \tag{67}$$

One can see from the above that if we make some constraints on the parameters of the system, to some certain type of nonlinear equation such as in this paper, we can get some exact solution of the equation with the simple combination of the hyperbolic function.

One direct extension of this method is to apply it to the type of coupled nonlinear equations that always appear in real systems [4–6]. In a subsequent paper, we shall deal with some coupled nonlinear equations, and find that the solutions of the coupled nonlinear equations are richer than those for a single equation.

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